Pricing Options and Equity-Indexed Annuities in a Regime-switching Model by Trinomial Tree Method

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This is a joint work with Kevin F. L. Yuen
Introduction

- Option pricing under Black-Scholes (Geometric Brownian motion) model has been studied extensively
- Closed form solution
- PDE
- Monte Carlo method
- Backward SDE
- Tree method
- Black-Scholes model is not perfect (or not correct)
- The main problem is the constant parameter assumption
- Markov Regime Switching Model (MRSM) provides an way to improve the model
Markov Regime-Switching Model (MRSM)

Markov regime-switching model

- Introduced by J. D. Hamilton (1989)
- Can reflect the information of the market environment which cannot be modeled by Gaussian process
- The parameters of the market model are controlled by a Markovian process
- Many results similar to those in Black-Scholes model can be obtained
Equity-indexed annuity (EIA) is a popular annuity product which
- is linked to the performance of an equity index (e.g. S&P 500)
- with a minimum rate of return
- might have a ceiling on the overall return
Trinomial Tree Model

Trinomial tree of Boyle (1986) is flexible. The extra branch of the trinomial tree gives one degree of freedom to the lattice and that makes it useful in regime switching model. A multi-state trinomial tree will be introduced which can be used to price options efficiently in multi-state model.
The model setting follows that in Buffington and Elliott (2002). Given a probability space \((\Omega, \mathcal{F}, P)\),

- \(P\) is the real-world probability measure
- \(\mathcal{T}\) denotes the time interval \([0, T]\)
- \(\{W(t)\}_{t \in \mathcal{T}}\) is a standard Brownian motion on \((\Omega, \mathcal{F}, P)\)
- \(\{X(t)\}_{t \in \mathcal{T}}\) is a continuous time Markov Chain process with finite state space \(\mathcal{X} := (e_1, e_2, \ldots, e_k)\),
- \(\{e_1, e_2, \ldots, e_k\}\), where \(e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^k\), represent the economic conditions
MRSM

We assume that expected rate of return, risk free interest rate and the volatility of the underlying asset depend only on the state of economic,

- risk free interest rate: $r_t = r(X(t)) = \langle \vec{r}, X(t) \rangle$
- expected return of stock: $\mu_t = \mu(X(t)) = \langle \vec{\mu}, X(t) \rangle$
- volatility of stock: $\sigma_t = \sigma(X(t)) = \langle \vec{\sigma}, X(t) \rangle$
One-State Trinomial Lattice

Binomial tree model
- Ratio of change of stock price is given by $e^{\sigma \sqrt{\Delta t}}$ and $e^{-\sigma \sqrt{\Delta t}}$ in CRR model

One-state trinomial tree model
- Stock price can remain the same, go up or go down by $e^{\lambda \sigma \sqrt{\Delta t}}$, $\lambda > 1$
If \( \pi_u, \pi_m, \pi_d \) are the risk neutral probabilities of the stock price increases, remains the same and decreases in the branch of the tree, \( \Delta t \) is the size of time step in the model, \( r \) is the risk free interest rate, then,

\[
\pi_u e^{\lambda \sigma \sqrt{\Delta t}} + \pi_m + \pi_d e^{-\lambda \sigma \sqrt{\Delta t}} = e^{r \Delta t} \quad \text{and} \quad \\
(\pi_u + \pi_d) \lambda^2 \sigma^2 \Delta t = \sigma^2 \Delta t \quad \text{(CRR)}
\]
λ should be larger than 1 so that the risk neutral probability measure exists. Usually, λ is taken to be $\sqrt{3}$ (Figlewski and Gao, 1999; Baule and Wilkens, 2004) or $\sqrt{1.5}$ (Boyle, 1988; Kamrad and Ritchken, 1991). By taking a fixed value of λ, the risk neutral measures can be found and the whole lattice can be constructed.
In the multi-state RMS, the risk free interest rate and the volatility is not a constant. They change according to the Markovian process. More branches can be introduced into the lattice so that the extra information can be incorporated into the model, for example, Boyle and Tian (1988), Kamrad and Ritchken (1991) and Bollen (1998). That will make the lattice structure much more complicated and the tree is no longer recombined.
Rather than increasing the branches in the tree, we change the risk neutral probability according to different regimes. The method relies on the flexibility of the trinomial tree model.
Modified Trinomial Lattice

Assuming that there are $k$ states in the Markovian regime switching model,

- Risk free interest rate: $r_1, r_2, \ldots, r_k$
- Volatility: $\sigma_1, \sigma_2, \ldots, \sigma_k$

The up-jump ratio of the lattice is taken to be $e^{\sigma \sqrt{\Delta t}}$. For a lattice which can be used for all regimes,

$$
\sigma > \max_{1 \leq i \leq k} \sigma_i
$$
For the regime $i$, $\pi^i_u, \pi^i_m, \pi^i_d$ are the risk neutral probabilities of the stock price increases, remains the same and decreases in the branch of the tree. Then, similar to the simple trinomial tree model, the following set of equations can be used, for each $1 \leq i \leq k$:

$$\pi^i_u e^{\sigma \sqrt{\Delta t}} + \pi^i_m + \pi^i_d e^{-\sigma \sqrt{\Delta t}} = e^{r_i \Delta t}$$

and

$$(\pi^i_u + \pi^i_d) \sigma^2 \Delta t = \sigma^2_i \Delta t$$
If $\lambda_i$ is defined to be $\sigma_i / \sigma$ for each $i$, then, $\lambda_i > 1$ and the value of $\pi_u^i, \pi_m^i, \pi_d^i$ can be obtained in terms of $\lambda_i$:

\[
\pi_u^i = \frac{e^{r_i \Delta t} - e^{-\sigma \sqrt{\Delta t}} - (1 - 1/\lambda_i^2)(1 - e^{-\sigma \sqrt{\Delta t}})}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}
\]

\[
\pi_m^i = 1 - \frac{\sigma_i^2}{\sigma^2} = 1 - \frac{1}{\lambda_i^2}
\]

\[
\pi_d^i = \frac{e^{\sigma \sqrt{\Delta t}} - e^{r_i \Delta t} - (1 - 1/\lambda_i^2)(e^{\sigma \sqrt{\Delta t}} - 1)}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}
\]
Modified Trinomial Lattice

Risk neutral probabilities depends on the value of $\sigma$. The choice of $\sigma$:

- Larger than all $\sigma_i$;
- Enhance the speed of convergence

We will take

$$\sigma = \max_{1 \leq i \leq k} \sigma_i + (\sqrt{1.5} - 1) \bar{\sigma}_i$$
Modified Trinomial Lattice

- $T$ is the expiration time
- $N$ is the number of time steps
- $\Delta t = \frac{T}{N}$ is the length of time step
- At time step $t$, there are $2t + 1$ nodes in the lattice, the node is counted from the lowest stock price level
- $S_{t,n}$ denotes the stock price of the $n^{th}$ node at time step $t$
- $V_{t,n,j}$ is the value of the derivative at the $n^{th}$ node at time step $t$ in the $j^{th}$ regime state
The transition probability of the Markovian process can be found by the generator matrix. If the generator matrix is assumed to be constant and denoted by $A$, the transition probability matrix, denoted by $P$, can be found by the following equation:

$$P(\Delta t) = \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix} = e^{A\Delta t} = I + \sum_{m=1}^{\infty} (\Delta t)^m A^m / m!$$
We start from the expiration time, for example, for an European call option with strike price $K$,

$$V_{N,n,i} = (S_{N,n} - K)^+$$

for all states $i$

where $S_{N,n} = S_0 \exp[(n - 1 - N)\sigma \sqrt{\Delta t}]$.

The Markovian process is independent of the Brownian motion and thus the transition probabilities will not be affected by changing the probability measure.
Using the following recursive equation:

\[ V_{t,n,i} = e^{-r_i \Delta t} \sum_{j=1}^{k} p_{ij} \left( \pi_u V_{t+1,n+2,j} + \pi_m V_{t+1,n+1,j} + \pi_d V_{t+1,n,j} \right) \]

the price of the option in all regimes (at any time) can be obtained.
American option

- Value of the option at each node under different regimes can be compared with the payoff of exercising the option.
- The larger value will be used as the price for iteration.

Barrier option

- Whole lattice is started from the lower barrier.
- Quadratic approximation will be used to calculate the price of the option using prices on the node.

Double barrier option

- Adjust the value of $\sigma$ so that both the upper and lower barrier will be on the grid.
The price of the following types of option will be calculated using multi-state trinomial tree:

- European option
- American option
- Barrier option
- Asian option

The model are tested by comparing with values in Boyle and Draviam (2007) to see if the method is really applicable.
**Table:** Comparison of different methods in pricing European call option in MRSM

<table>
<thead>
<tr>
<th>European Call Option $i$</th>
<th>Regime 1</th>
<th>Regime 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>Naik</td>
<td>B &amp; D</td>
</tr>
<tr>
<td>94</td>
<td>5.8620</td>
<td>5.8579</td>
</tr>
<tr>
<td>98</td>
<td>8.0844</td>
<td>8.0775</td>
</tr>
</tbody>
</table>
## Table: Comparison of different methods in pricing European call option in MRSM

<table>
<thead>
<tr>
<th>European Call Option $K$</th>
<th>Regime 1</th>
<th>Regime 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>Naik</td>
<td>B &amp; D</td>
</tr>
<tr>
<td>94</td>
<td>6.2748</td>
<td>6.2705</td>
</tr>
<tr>
<td>96</td>
<td>7.3408</td>
<td>7.3352</td>
</tr>
<tr>
<td>102</td>
<td>11.0820</td>
<td>11.0755</td>
</tr>
<tr>
<td>106</td>
<td>13.9777</td>
<td>13.9726</td>
</tr>
</tbody>
</table>
Model Assumptions

- Two-regime
- Risk free rate: (.04, .06)
- Volatility: (.25, .35)
- Initial price and strike price: 100
- Generator matrix:

\[ \eta = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} \]
### Table: Pricing the American put option with the trinomial tree

<table>
<thead>
<tr>
<th></th>
<th>Regime 1</th>
<th>Regime 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>Price</td>
</tr>
<tr>
<td>20</td>
<td>8.80315</td>
<td>0.05236</td>
</tr>
<tr>
<td>40</td>
<td>8.85551</td>
<td>0.02674</td>
</tr>
<tr>
<td>80</td>
<td>8.88225</td>
<td>0.01300</td>
</tr>
<tr>
<td>160</td>
<td>8.89525</td>
<td>0.00633</td>
</tr>
<tr>
<td>320</td>
<td>8.90158</td>
<td>0.00313</td>
</tr>
<tr>
<td>640</td>
<td>8.90471</td>
<td>0.00156</td>
</tr>
<tr>
<td>1280</td>
<td>8.90627</td>
<td>0.00077</td>
</tr>
<tr>
<td>2560</td>
<td>8.90704</td>
<td>0.00038</td>
</tr>
<tr>
<td>5120</td>
<td>8.90742</td>
<td></td>
</tr>
</tbody>
</table>
Table: Pricing the down-and-out barrier call option with the trinomial tree (The barrier level is set as 90)

<table>
<thead>
<tr>
<th>N</th>
<th>Price</th>
<th>Diff</th>
<th>Ratio</th>
<th>Price</th>
<th>Diff</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>8.97860</td>
<td>-0.01239</td>
<td>-0.6917</td>
<td>9.73967</td>
<td>-0.02790</td>
<td>0.0487</td>
</tr>
<tr>
<td>40</td>
<td>8.96621</td>
<td>0.00857</td>
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<td>9.71177</td>
<td>-0.00136</td>
<td>4.9779</td>
</tr>
<tr>
<td>80</td>
<td>8.97478</td>
<td>-0.00414</td>
<td>0.1304</td>
<td>9.71041</td>
<td>-0.00677</td>
<td>0.3840</td>
</tr>
<tr>
<td>160</td>
<td>8.97064</td>
<td>-0.00054</td>
<td>-0.2778</td>
<td>9.70364</td>
<td>-0.00260</td>
<td>0.3269</td>
</tr>
<tr>
<td>320</td>
<td>8.97010</td>
<td>0.00015</td>
<td>-6.4667</td>
<td>9.70104</td>
<td>-0.00085</td>
<td>1.2588</td>
</tr>
<tr>
<td>640</td>
<td>8.97025</td>
<td>-0.00097</td>
<td>-0.3505</td>
<td>9.70019</td>
<td>-0.00107</td>
<td>0.0748</td>
</tr>
<tr>
<td>1280</td>
<td>8.96928</td>
<td>0.00034</td>
<td>-0.2059</td>
<td>9.69912</td>
<td>-0.00008</td>
<td>2.1250</td>
</tr>
<tr>
<td>2560</td>
<td>8.96962</td>
<td>-0.00007</td>
<td></td>
<td>9.69904</td>
<td>-0.00017</td>
<td></td>
</tr>
<tr>
<td>5120</td>
<td>8.96955</td>
<td></td>
<td></td>
<td>9.69887</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table: Pricing the double barrier call option with the trinomial tree

<table>
<thead>
<tr>
<th>N</th>
<th>Price (Regime 1)</th>
<th>Diff (Regime 1)</th>
<th>Ratio (Regime 1)</th>
<th>Price (Regime 2)</th>
<th>Diff (Regime 2)</th>
<th>Ratio (Regime 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>6.15869</td>
<td>-0.15826</td>
<td>0.7097</td>
<td>4.54096</td>
<td>-0.13822</td>
<td>0.6130</td>
</tr>
<tr>
<td>40</td>
<td>6.00043</td>
<td>-0.11232</td>
<td>0.7314</td>
<td>4.40274</td>
<td>-0.08473</td>
<td>0.5189</td>
</tr>
<tr>
<td>80</td>
<td>5.88811</td>
<td>-0.04845</td>
<td>0.4111</td>
<td>4.31801</td>
<td>-0.04397</td>
<td>0.3834</td>
</tr>
<tr>
<td>160</td>
<td>5.83966</td>
<td>-0.01992</td>
<td>0.5954</td>
<td>4.27404</td>
<td>-0.01686</td>
<td>0.6109</td>
</tr>
<tr>
<td>320</td>
<td>5.81974</td>
<td>-0.01186</td>
<td>0.5320</td>
<td>4.25718</td>
<td>-0.01030</td>
<td>0.5029</td>
</tr>
<tr>
<td>640</td>
<td>5.80788</td>
<td>-0.00631</td>
<td>0.6133</td>
<td>4.24688</td>
<td>-0.00518</td>
<td>0.6120</td>
</tr>
<tr>
<td>1280</td>
<td>5.80157</td>
<td>-0.00387</td>
<td>0.1731</td>
<td>4.24170</td>
<td>-0.00317</td>
<td>0.2145</td>
</tr>
<tr>
<td>2560</td>
<td>5.79770</td>
<td>-0.00067</td>
<td></td>
<td>4.23853</td>
<td>-0.00068</td>
<td></td>
</tr>
<tr>
<td>5120</td>
<td>5.79703</td>
<td></td>
<td></td>
<td>4.23785</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The barrier level is set as 70 and 150, respectively.
### Table: Price of the double barrier call options with different barrier levels

<table>
<thead>
<tr>
<th>Double Barrier Call Option in Regime 1</th>
<th>90</th>
<th>80</th>
<th>70</th>
<th>60</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>0.00063</td>
<td>0.0249</td>
<td>0.0498</td>
<td>0.0544</td>
<td>0.0546</td>
</tr>
<tr>
<td>120</td>
<td>0.10229</td>
<td>0.4310</td>
<td>0.5773</td>
<td>0.5952</td>
<td>0.5970</td>
</tr>
<tr>
<td>130</td>
<td>0.71002</td>
<td>1.6257</td>
<td>1.9120</td>
<td>1.9422</td>
<td>1.9451</td>
</tr>
<tr>
<td>140</td>
<td>1.88418</td>
<td>3.4101</td>
<td>3.8049</td>
<td>3.8446</td>
<td>3.8463</td>
</tr>
<tr>
<td>150</td>
<td>3.30481</td>
<td>5.3336</td>
<td>5.8019</td>
<td>5.8474</td>
<td>5.8490</td>
</tr>
<tr>
<td>200</td>
<td>7.87455</td>
<td>10.8888</td>
<td>11.4649</td>
<td>11.5163</td>
<td>11.5183</td>
</tr>
</tbody>
</table>
Double Barrier Call Option in Regime 2

<table>
<thead>
<tr>
<th></th>
<th>90</th>
<th>80</th>
<th>70</th>
<th>60</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>0.00004</td>
<td>0.0049</td>
<td>0.0202</td>
<td>0.0285</td>
<td>0.0297</td>
</tr>
<tr>
<td>120</td>
<td>0.01567</td>
<td>0.1446</td>
<td>0.2909</td>
<td>0.3385</td>
<td>0.3440</td>
</tr>
<tr>
<td>130</td>
<td>0.01933</td>
<td>0.7381</td>
<td>1.1160</td>
<td>1.2117</td>
<td>1.2210</td>
</tr>
<tr>
<td>140</td>
<td>0.73257</td>
<td>1.8882</td>
<td>2.5051</td>
<td>2.6410</td>
<td>2.6515</td>
</tr>
<tr>
<td>150</td>
<td>1.62095</td>
<td>3.4224</td>
<td>4.2422</td>
<td>4.4065</td>
<td>4.4181</td>
</tr>
<tr>
<td>200</td>
<td>6.65432</td>
<td>10.5198</td>
<td>11.7835</td>
<td>11.9909</td>
<td>12.0042</td>
</tr>
</tbody>
</table>

The price of the double barrier options with lower barriers of 90, 80, 70, 60, 50 and upper barriers of 110, 120, 130, 140, 150, 200 in the two regimes are calculated using 1000 time steps.
European option:
- Convergence order is the same as that in CRR model
- Regime 2 has a smaller error
American option:

- Prices of American call option found by the modified trinomial model is the same as the European call option.
- Early exercise is not optimal if there is no dividend being distributed.
- Rate of convergence for the regime 2 is very fast, even faster than that of European put option.
- Put option can be exercised earlier.
Numerical Results and Analysis

Down-and-out barrier option:
- Cheaper than those of the European call option due to the presence of the down-and-out barrier
- Prices in the two regimes are closer compare with the case of European option
- Higher volatility of regime 2 has a higher chance to achieve a higher value and hitting the down-and-out barrier at expiration
- Convergence pattern of barrier option is more complicated
Double barrier option:

- Lattice is built from the lower barrier and touched the upper barrier by controlling the value of $\sigma$ used in the lattice.
- Increase the separation of barriers makes the price of double barrier option converges to vanilla option.
Pricing Asian Option

At the $n^{th}$ node of time step $t$ under the $j^{th}$ regime state, let
- $V_{t,n,m,j}$ be the value of the derivative with representative value $m$
- $Va(t, n, s, j)$ be the value of the derivative with average index value $s$

We start our calculation from the expiration time at representative level, for example, for an average price call option with strike $K$, 

$$V_{T,n,m,i} = Va(T, n, S_0 e^{mh}, i) = (S_0 e^{mh} - K)^+$$ for all states $i$ and all nodes $n$. 

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Pricing Asian Option

For the option on each representative value of nodes at previous time points, we have

\[
V_{t,n,m,i} = V_a(t, n, S_0 e^{m h}, i)
\]

\[
e^{-r_i \Delta t} \sum_{j=1}^{k} p_{ij} [\pi^i u V_a(t + 1, n + 2, \frac{tS_0 e^{mh} + uS_{t,n}}{t + 1}, j)]
\]

\[
+ \pi^i m V_a(t + 1, n + 1, \frac{tS_0 e^{mh} + S_{t,n}}{t + 1}, j)
\]

\[
+ \pi^i d V_a(t + 1, n, \frac{tS_0 e^{mh} + dS_{t,n}}{t + 1}, j)],
\]
Pricing Asian Option

Average price might not be at the representative level, we further use linear approximation, if we have $m = \lfloor \ln(\text{AVE}) / h \rfloor$ where AVE denotes the average stock price value, then,

$$
V_a(t, n, \text{AVE}, i) = \frac{\text{AVE} - S_0 e^{mh}}{S_0 e^{mh}(e^h - 1)} V_a(t, n, S_0 e^{(m+1)h}, j) + \frac{S_0 e^{(m+1)h} - \text{AVE}}{S_0 e^{mh}(e^h - 1)} V_a(t, n, S_0 e^{mh}, j)
$$

$$
= \frac{\text{AVE} - S_0 e^{mh}}{S_0 e^{mh}(e^h - 1)} V_{t,n,m+1,j} + \frac{S_0 e^{(m+1)h} - \text{AVE}}{S_0 e^{mh}(e^h - 1)} V_{t,n,m,j}
$$

Using these equations recursively, we can obtain the fair value of the Asian option.
Pricing EIA

Let $A(t)$ be the average index level from 0 to $t$, that is,

$$A(t) = \frac{1}{t} \int_0^t S(u) du$$

Then, we consider the general expression of a point-to-point Asian EIA which has cumulative return equal to

$$C(t) = \max \left[ \min [1 + \alpha R_t, (1 + \zeta)^t], (1 + g)^t \right]$$

where $R_t = \frac{A(t)}{S_0} - 1$, $\alpha$ is the participation rate, $\zeta$ is the cap rate and $g$ is the guarantee rate of the EIA contract.
If we focus on the cumulative return of the EIA, we would find that it is very similar to a collar:

\[
\begin{align*}
\max &\left[ \min \left[ 1 + \alpha R_t, (1 + \zeta)^t \right], (1 + g)^t \right] \\
= &\ \max \left[ 1 + \alpha R_t - \left[1 + \alpha R_t - (1 + \zeta)^t \right]^+, (1 + g)^t \right] \\
= &\ 1 + \alpha R_t - \left[1 + \alpha R_t - (1 + \zeta)^t \right]^+ + \\
&\ [(1 + g)^t - (1 + \alpha R_t) + \left[1 + \alpha R_t - (1 + \zeta)^t \right]^+]^+ \\
= &\ 1 + \alpha R_t - \left[1 + \alpha R_t - (1 + \zeta)^t \right]^+ + [(1 + g)^t - (1 + \alpha R_t)]^+. 
\end{align*}
\]
Furthermore, we have

\[
[1 + \alpha R_t - (1 + \zeta)^t]^+ = \frac{\alpha}{S_0} \left[ \frac{S_0}{\alpha} + A(t) - S_0 - \frac{S_0}{\alpha} (1 + \zeta)^t \right]^+
\]

\[
= \frac{\alpha}{S_0} \left[ A(t) - S_0 (1 + \frac{(1+\zeta)^t - 1}{\alpha}) \right]^+
\]

and

\[
[(1 + g)^t - (1 + \alpha R_t)]^+ = \frac{\alpha}{S_0} \left[ S_0 (1 + \frac{(1+g)^t - 1}{\alpha}) - A(t) \right]^+.
\]

We can find the return of EIA in terms of standard Asian options.
Introduction
Multi-state Trinomial Tree Model
Pricing Asian Option
Numerical Results and Analysis

Pricing Ratchet EIA

Ratchet EIA allows the investors to lock the annual return into their account. Its cumulative return is

\[ C(t) = \prod_{k=1}^{t} \max[min(1 + \alpha R'_k, 1 + \zeta), 1 + g]. \]

where \( R'_k = \int_{k-1}^{k} S(u)du/S(k - 1) - 1 \), which is the average index return of the \( k^{th} \) year.
Pricing Ratchet EIA

In simple Black Scholes assumption,

- the appreciation rate of the index is independent of the previous return due to the Markov property of Brownian motion
- expected return of the whole contract period is equal to the product of expected return in each year

In Markov Regime Switching Model,

- future return rate and volatility of the index are affected by the current data due to the presence of regime switching
- able to solve this problem by considering a conditional expectation due to the Markov property of the regime switching process.
Pricing Ratchet EIA

The expected risk neutral discounted value of the unit ratchet EIA for $t$ years with initial regime $i$ is defined as $V_r(t, i)$ which is

$$
E^Q \left[ \exp \left( - \int_0^t r(u) du \right) \prod_{k=1}^t \max \left[ \min \left( 1 + \alpha R_k', 1 + \zeta \right), \ 1 + g \right] X(0) = x_i \right]
$$

$$
= E^Q \left[ \exp \left( - \int_0^1 r(u) du \right) \max \left[ \min \left( 1 + \alpha R_1', 1 + \zeta \right), \ 1 + g \right] \left. E^Q \left[ \exp \left( - \int_1^t r(u) du \right) \prod_{k=2}^t \max \left[ \min \left( 1 + \alpha R_k', 1 + \zeta \right), \ 1 + g \right] G_1, X(0) = x_i \right| X(0) = x_i \right]
$$

$$
= V_r(1, i) \sum_{j_1=1}^k p_{ij_1}(1) V_r(t - 1, j_1)
$$

With price of one-year simple EIA, we can obtain the price of unit ratchet EIA.
If the ratchet EIA is payable at the end of the year that

- the investor dies or
- the expiration of the EIA contract.

We assume that the future lifetime, the probability of death, survival in one year of $x$-year-old people are $T(x)$, $q_x$ and $p_x$ respectively, and there is no selection effect and the lifetime is independent of the Brownian motion and regime-switching process.

Let $\{F_t^M\}_{t \in T}$ be the the natural filtration of the future lifetime random variable. We define $\mathcal{H}_t$ to be the $\sigma$-algebra $F_t^X \lor F_t^Z \lor F_t^M$. 
Pricing Ratchet EIA with mortality assumption

The expected risk neutral discounted value of the unit ratchet EIA for \( t \) years with initial regime \( i \) for a \( x \)-year old person is defined as \( V_{m,x}(t, i) \) which is equal to

\[
E^Q \left[ \exp[- \int_0^{t \wedge \lceil T(x) \rceil} r(u) du] \prod_{k=1}^{t \wedge \lceil T(x) \rceil} \max[\min(1 + \alpha R'_k, 1 + \zeta), 1 + g] | X(0) = x_i] \right]
\]

\[
= E^Q \left[ \exp[- \int_0^1 r(u) du] \max[\min(1 + \alpha R'_1, 1 + \zeta), 1 + g] \right]
\]

\[
\left[ I(T(x) \leq 1) + I(T(x) > 1) E^Q \left[ \exp[- \int_1^{t \wedge \lceil T(x+1) + 1 \rceil} r(u) du] \right. \right.
\]

\[
\left. \left. \prod_{k=2}^{t \wedge \lceil T(x+1) + 1 \rceil} \max[\min(1 + \alpha R'_k, 1 + \zeta), 1 + g] | \mathcal{H}_1, X(0) = x_i] \right) \right] | X(0) = x_i]
\]

\[
= V_{m,x}(1, i) \left[ q_x + p_x \sum_{j_1=1}^{k} p_{ij_1}(1) V_{m,x+1}(t - 1, j_1) \right].
\]
If we have the values of ratchet EIAs with different expirations, together with the mortality information, we have

\[ V_{m,x}(1, i) = V_r(1, i) \]  
for all \( i \) and all \( x \), therefore

\[ V_{m,x}(t, i) = \sum_{n=0}^{t-1} n p_x q_{x+n} V_r(n + 1, i) \]

where \( n p_x \) is the probability of a \( x \)-year-old person survives for the coming \( n \) years.
We compare our results with those of Hull and White (1993),
- initial asset price, strike price: 50
- volatility: 30%, risk free rate: 10%
- average price asian option

The option price obtained by using the trinomial lattice is very close to the value obtained by Hull and White (1993) and that means trinomial tree has a similar performance as binomial tree in pricing Asian options.
Non Regime Switching Model

Table: Comparison of the prices of average price Asian options in simple BS model

<table>
<thead>
<tr>
<th>Average Price Asian Option</th>
<th>Number of Time Steps (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20</td>
</tr>
<tr>
<td>h</td>
<td>Method</td>
</tr>
<tr>
<td></td>
<td>Tree</td>
</tr>
<tr>
<td>0.050</td>
<td>Tree</td>
</tr>
<tr>
<td>0.010</td>
<td>Tree</td>
</tr>
<tr>
<td>0.005</td>
<td>Tree</td>
</tr>
<tr>
<td>0.003</td>
<td>Tree</td>
</tr>
<tr>
<td></td>
<td>H&amp;W</td>
</tr>
</tbody>
</table>
Asian option in MRSM

We then compare our method with Boyle and Draviam (2007),

- one-year average price call option
- initial asset price: $S_0$ with 2 regimes
  - regime 0; volatility: 15%, risk free rate: 5%
  - regime 1; volatility: 25%, risk free rate: 5%
- generator matrix for the Markov chain: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. 
Asian option in MRSM

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>Method</th>
<th>Regime 0, $\sigma_0 = 0.15$, $r_0 = 0.05$</th>
<th>Regime 1, $\sigma_1 = 0.25$, $r_0 = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$K = 90$</td>
<td>$K = 100$</td>
</tr>
<tr>
<td>90</td>
<td>Tree</td>
<td>4.6104</td>
<td>1.1134</td>
</tr>
<tr>
<td></td>
<td>B&amp;D</td>
<td>4.6204</td>
<td>1.1172</td>
</tr>
<tr>
<td>95</td>
<td>Tree</td>
<td>8.1076</td>
<td>2.6203</td>
</tr>
<tr>
<td></td>
<td>B&amp;D</td>
<td>8.1132</td>
<td>2.6288</td>
</tr>
<tr>
<td>100</td>
<td>Tree</td>
<td>12.3354</td>
<td>5.1227</td>
</tr>
</tbody>
</table>
EIA in MRSM

We find the value of unit EIA with the same condition as above:

- one-year average price call option
- initial asset price: $S_0$ with 2 regimes
- regime 0; volatility: 15%, risk free rate: 5%
- regime 1; volatility: 25%, risk free rate: 5%
- participation rate: 1

- generator matrix for the Markov chain: $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. 

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EIA in MRSM

**Table:** Price of one-year unit EIA in MRSM

<table>
<thead>
<tr>
<th>Guarantee Return</th>
<th>Price of one-year EIA at Regime 0</th>
<th>Price of one-year EIA at Regime 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.96881 0.98304 0.99085 0.99476 0.99659</td>
<td>0.96076 0.97743 0.98878 0.99610 1.00062</td>
</tr>
<tr>
<td>1%</td>
<td>0.97292 0.98715 0.99496 0.99887 1.00070</td>
<td>0.96498 0.98166 0.99300 1.00032 1.00484</td>
</tr>
<tr>
<td>2%</td>
<td>0.97741 0.99164 0.99946 1.00336 1.00520</td>
<td>0.96947 0.98615 0.99749 1.00481 1.00934</td>
</tr>
<tr>
<td>3%</td>
<td>0.98229 0.99652 1.00434 1.00824 1.01008</td>
<td>0.97424 0.99092 1.00226 1.00958 1.01411</td>
</tr>
</tbody>
</table>
Ratchet EIA in MRSM

With the value of one year unit EIA, we would be able to obtain the price of ratchet EIAs easily using recursive equation. Assume

- ceiling return: 20%
- guarantee return: 3%

The price of unit annual reset EIA of different expirations are shown below.

Table: Price of unit annual reset EIA in MRSM

<table>
<thead>
<tr>
<th>Regime</th>
<th>Length of ratchet EIA (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1.00824</td>
</tr>
<tr>
<td>1</td>
<td>1.00958</td>
</tr>
</tbody>
</table>

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References


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Thank You.

Q & A
Thank You.

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